A CHARACTERIZATION OF COMMUTATIVITY FOR NON-ASSOCIATIVE NORMED ALGEBRAS

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0.- Introduction

A celebrated Theorem of C. LePage [**12**] reads as follows.

(1) If A is a complete normed associative complex algebra with a unit, and if there exists a positive constant k satisfying $\| yx \| \leq k \| xy \|$ for all x, y in A, then A is commutative.

Actually, minor changes on the proof of LePage's theorem allow to show the next more general result (see [**2**; Proposition 15.5]).

(2) Let A be a complete normed associative complex algebra with a unit e, and $\odot$ be a (possibly non-associative) product on A satisfying

1. *$x \odot e = x$ for every x in A, and*
2. *$\| x \odot y \| \leq k \| xy \|$ for some positive constant k and all x, y in A.*

Then $\odot$ coincides with the product of A.

In this paper we mainly deal with the natural question if the requirement of associativeness in Assertions (1) and (2) above can be removed. For any complex algebra A, denote by $N(A)$ the set of those elements x in A such that the spectrum of the operator of right multiplication by x is countable. As main result, we prove in Corollary 1.2 that associativeness can be actually removed in (2) (and hence also in (1)) whenever the linear hull of $N(A)$ is dense in A (for instance, whenever A is finite-dimensional). As an application of the main tool for the above result, we also prove that associativeness can be removed in (1) whenever A is either a nondegenerate non-commutative Jordan algebra with essential socle (Corollary 2.5) or a non-commutative JB^*-algebra (Corollary 3.5). In the last case, the existence of a unit for A is not required. A discussion about the methods of proof for the above results leads us to find that associativeness can be relaxed in (2) to right alternativeness (Proposition 4.4), and in (1) to split quasiassociativeness (Corollary 4.9).

The concluding section of the paper is devoted to associative algebras. We prove that the assumption in (1) that A has a unit can be drastically relaxed. In fact (see Corollaries 5.2 and 5.3) Assertion 1 remains true if that assumption is replaced by anyone of the following:

1. A^2 is dense in A.
2. A has zero annihilator.

1.- The main tool

Given a complex Banach space E and a bounded linear operator T on E, the Banach isomorphism theorem ensures that the spectrum of T relative to the Banach algebra $BL(E)$ of all bounded linear operators on E coincides with the spectrum of T relative to the algebra of all (possibly unbounded) linear operators on E. We simply denote it by $Sp(T)$. For an element x in an algebra A, the symbol R_x will stand for the operator of right multiplication by x on A. The main tool for our work is the following theorem.

THEOREM 1.1.- *Let A be a (possibly non-associative) complete normed complex algebra with a right unit e, E complex Banach space, $h : A \times A \to E$ a bilinear mapping satisfying $\| h(x,y) \| \leq k \| xy \|$ for some positive constant k and all x, y in A, and z be in A such that $Sp(R_z)$ is countable. Then, for every x in A, we have $h(x,z) = h(xz,e)$.*

Proof.- Replacing z by $z - \alpha e$, for a suitable α in $\mathbb{C}$, we can assume that $o \notin Sp(R_z)$. Then $K := \{\mu^{-1} \ : \ \mu \in Sp(R_z)\}$ is a countable compact subset of $\mathbb{C}$. Putting $\Omega := \mathbb{C} \backslash K$, and considering the analytic mapping $\varphi : \Omega \to A$ given by $\varphi(\lambda) := e - \lambda z$, we realize that, for every λ in Ω, the operator $R_{\varphi(\lambda)}$ is bijective. If for y in A we denote by T_y the continuous linear mapping from A to E defined by $T_y(a) := h(a,y)$, then the assumption on h leads to the inequality $\| T_{\varphi(\lambda)}(a) \| \leq k \| R_{\varphi(\lambda)}(a) \|$ for all λ in Ω and A in A. Equivalently, we have $\| T_{\varphi(\lambda)} \circ R^{-1}_{\varphi(\lambda)}(x) \| \leq k \| x \|$ for all λ in Ω and x in A, and hence $\| T_{\varphi(\lambda)} \circ R^{-1}_{\varphi(\lambda)} \| \leq k$ for every λ in Ω. Now, let us fix an arbitrary continuous linear functional f on the complex Banach space $BL(A,E)$ of all bounded linear mappings from A into E. Then the function $\Psi : \lambda \to f(T_{\varphi(\lambda)} \circ R^{-1}_{\varphi(\lambda)})$ from Ω to $\mathbb{C}$ is analytic and bounded. Since Ω is the complement in $\mathbb{C}$ of a countable compact set, it follows from an extended version of Liouville's theorem (see for instance [**20**; Exercise 10.(a), p. 324]) that Ψ is constant. As a consequence, we have $f(T_e \circ R_z - T_z) = \Psi'(0) = 0$. Since f is arbitrary in the dual of

$BL(A,E)$, the Hahn-Banach theorem yields the equality $T_z = T_e \circ R_z$, that is, $h(x,z) = h(xz,e)$ for every x in A. ■

Let E be a vector space. By a product on E we mean any bilinear mapping $(x,y) \to x \odot y$ from $E \times E$ into E. Given a product $\odot$ on E and an element u in E, we say that $\odot$ is right u-admissible if the equality $x \odot u = x$ holds for every x in E. The next result is a direct consequence of Theorem 1.1.

COROLLARY 1.2.- *Let A be a complete normed complex algebra with a right unit e, and $\odot$ be a right e-admissible product on (the vector space of) A satisfying $\| x \odot y \| \leq k \| xy \|$ for some positive constant k and all x, y in A. If the linear hull of the set*

$$\{z \in A \ : \ Sp(R_z) \text{ is countable}\}$$

is dense in A (for instance, if A is finite-dimensional), then $\odot$ coincides with the canonical product of A.

In the case that e is in fact a (two-sided) unit for the algebra A, the product $\odot$ on A defined by $x \odot y := yx$ is right e-admissible, hence Corollary 1.2 applies to get the next variant of LePage's theorem.

COROLLARY 1.3.- *Let A be a complete normed complex algebra with a unit. If the linear hull of the set*

$$\{z \in A \ : \ Sp(R_z) \text{ is countable}\}$$

is dense in A, and if there exists $k > 0$ such that $\| yx \| \leq k \| xy \|$ for all x, y in A, then A is commutative.

2.- Applications to Jordan algebras

Jordan algebras are defined as those commutative algebras satisfying the " Jordan identity " $(x^2y)x = x^2(yx)$. Let A be a Jordan complex algebra with a unit E. An element x in A is said to be invertible in A if there exists y in A satisfying $xy = e$ and $x^2y = x$. The spectrum $Sp(A,z)$ of an arbitrary element z in A is defined by the equality

$$Sp(A,z) := \{\lambda \in \mathbb{C} : z - \lambda e \text{ is not invertible in } A\}.$$

When A is complete normed we know that, for every z in A, the inclusion

$$Sp(R_z) \ \subseteq \ \frac{1}{2} \ (Sp(A,z) \ + \ Sp(A,z))$$

holds [**13**; Theorem 1.2]. Now, the next result follows from Theorem 1.1.

COROLLARY 2.1.- *Let A be a complete normed Jordan complex algebra with a unit e, E a complex Banach space, $h : A \times A \to E$ a bilinear mapping satisfying $\| h(x,y) \| \leq k \| xy \|$ for some positive constant k and all x, y in A, and z be in A such that $Sp(A,z)$ is countable. Then, for every x in A, we have $h(x,z) = h(xz,e)$.*

As a consequence, if A is a complete normed Jordan complex algebra with a unit e, if $\odot$ is a right e-admissible product on A satisfying $\| x \odot y \| \leq k \| xy \|$ for some positive constant k and all x, y in A, and if the linear hull of the set

$$\{z \in A \ : \ Sp(A,z) \text{ is countable}\}$$

is dense in A, then $\odot$ coincides with the canonical product of A.

For every algebra A, let us denote by A^+ the algebra consisting of the vector space of A and the product $x.y := \frac{1}{2}(xy + yx)$. The algebra A is said to be Jordan admissible if A^+ is a Jordan algebra. An element x in a Jordan admissible complex algebra A with a unit is said to be invertible in A if it is invertible in A^+. Consequently, for arbitrary z in such an algebra A, we put $Sp(A,z) := Sp(A^+, z)$. The convention just established has its roots in the fact that, if A is an associative algebra with a unit, then the invertible elements in A (in the usual associative meaning) are nothing but the invertible elements in the Jordan algebra A^+ (in the sense provided at the beginning of this section) [**10**; page 51]. Recall that, given a bounded linear operator T on a complex Banach space E, the approximate point spectrum of T, $\sigma_{ap}(T)$, is defined as the set of those complex numbers λ such that there is a sequence $\{x_n\}$ of norm-one elements in E satisfying $\lim_{n\to\infty} \| \lambda x_n - T(x_n) \| = 0$. According to [**1**; Theorem 57.7], $\sigma_{ap}(T)$ contains the boundary of $Sp(T)$.

THEOREM 2.2.- *Let A be a complete normed Jordan admissible complex algebra with a unit, and z be in A such that $Sp(A,z)$ is countable. Assume that there exists $k > 0$ satisfying $\| yx \| \leq k \| xy \|$ for all x, y in A. Then z commutes with every element in A.*

Proof.- Let R_z^+ denote the operator of multiplication by z on A^+. Then, for every x in A and all complex numbers λ, we have

$$\| \lambda x - R_z^+(x) \| \ = \ \| \lambda x - \frac{1}{2}(xz + zx) \|$$

$$\leq \ \| \frac{1}{2}(\lambda x - xz) \| \ + \ \| \frac{1}{2}(\lambda x - zx) \| \ \leq \ \frac{1}{2}(1+k) \| \lambda x - R_z(x) \| \ .$$

Therefore $\sigma_{ap}(R_z)$ (and hence the boundary of $Sp(R_z)$) is contained in $Sp(R_z^+)$. Since $Sp(A,z)$ $(= Sp(A^+,z))$ is countable, it follows from the already known inclusion

$$Sp(R_z^+) \ \subseteq \ \frac{1}{2} \ (Sp(A^+,z) \ + \ Sp(A^+,z))$$

that $Sp(R_z)$ is countable too. Now, we can apply Theorem 1.1, with $E := A$ and $h(x,y) := yx$ for all x, y in A, to obtain that z commutes with every element in A. ■

An algebra A is said to be quadratic if it has a unit e and, for every x in A, the subalgebra of A generated by $\{e, x\}$ has dimension at most two. If A is a quadratic complex algebra, then A is Jordan admissible and every element in A has a finite spectrum. Therefore we have:

COROLLARY 2.3.- *Let A be a complete normed quadratic complex algebra such that there exists $k > 0$ satisfying $\| yx \| \leq k \| xy \|$ for all x, y in A. Then A is commutative.*

Among Jordan admissible algebras, the so called non-commutative Jordan algebras become specially relevant. Non-commutative Jordan algebras can be defined as those Jordan admissible algebras satisfying the " flexibility condition " $(xy)x = x(yx)$ [**21**; p. 141]. Let A be a flexible algebra. Then, for every a in A, the mapping $x \to ax - xa$ is a derivation of A^+ [**21**; p.146], and therefore the set

$$Z := \{z \in A \ : \ z \text{ commutes with every element in } A\}$$

is a subalgebra of A^+. Since Z is a commutative subset of A, it is in fact a subalgebra of A. Now, Theorem 2.2 leads to the following corollary.

COROLLARY 2.4.- *Let A be a complete normed non-commutative Jordan complex algebra with a unit. If the subalgebra of A generated by the set*

$$\{z \in A \ : \ Sp(A, z) \text{ is countable}\}$$

is dense in A, and if there exists $k > 0$ such that $\| yx \| \leq k \| xy \|$ for all x, y in A, then A is commutative.

Let A be a non-commutative Jordan algebra. For x in A, we denote by U_x the operator on A given by $U_x(y) := x(xy + yx) - x^2y$ for all y in A. For later application we note that, if x is in A, then the equality $U_x = U_x^+$ holds, where U_x^+ means the U_x-operator relative to the algebra A^+. A is said to be nondegenerate if $U_x = 0$ implies $x = 0$. Vector subspaces I of A satisfying $U_I(A) \subseteq I$ are called inner ideals of A. The socle of A is defined as the sum of all minimal inner ideals of A. If A is nondegenerate, then the socle of A is a (two-sided) ideal of A ([**14**], [**6**]).

COROLLARY 2.5.- *Let A be a complete normed nondegenerate non-commutative Jordan complex algebra with a unit and essential socle. Assume that there exists $k > 0$ satisfying $\| yx \| \leq k \| xy \|$ for all x, y in A. Then A is commutative.*

Proof.- By [**7**; Theorem 1], every element in the socle of A has a finite spectrum. Then, by Theorem 2.2, the socle of A is commutative. Therefore, by [**5**; Corollary 7], A is commutative. ■

3.- The case of non-commutative JB*-algebras

A celebrated theorem of I. Kaplansky [**11**; Appendix III, Theorem B] asserts that a C^*-algebra A is commutative if (and only if) A has no non zero elements z with $z^2 = 0$. This criterion of commutativity for C^*-algebras is very powerful, as shown in particular by the next LePage-type application.

OBSERVATION 3.1.- *Let A be a (possibly non unital) C^*-algebra such that there exists $k > 0$ satisfying $\| yx \| \leq k \| xy \|$ for all x, y in A. Then A is commutative.*

Proof.- Let z be in A satisfying $z^2 = 0$. Then we have

$$\begin{aligned} \| z \|^6 \; = \; \| z^*z \|^3 \; = \; \| (z^*z)^3 \| \; = \; \| z^*zz^*zz^*z \| \; = \; \| (zz^*z)^*(zz^*z) \| \\ = \; \| zz^*z \|^2 \; \leq \; k^2 \| z^*z^2 \|^2 \; = \; 0 \; , \end{aligned}$$

and therefore $z = 0$. By Kaplansky's theorem, A is commutative. ■

The main aim in this section is to prove that the result in Observation 3.1 remains true if we relax the assumption that A is a C^*-algebra to the one that A is a non-commutative JB^*-algebra. Non-commutative JB^*-algebras are defined as those complete normed non-commutative Jordan complex algebras A with a conjugate-linear algebra involution $*$ satisfying $\| U_x(x^*) \| = \| x \|^3$ for every x in A. Non-commutative JB^*-algebras arise in a natural way in Functional Analysis. Indeed, if a norm-unital complete normed non-associative complex algebra A is subjected to the geometric Vidav condition characterizing C^*-algebras in the associative context [**2**; Theorem 38.14], then A is a non-commutative JB^*-algebra [**18**]. Let A be a non-commutative JB^*-algebra. It is known that the set $Symm(A)$ of all *-invariant elements of A (regarded in the natural way as a closed real subalgebra of A^+) is a JB-algebra [**8**; Proposition 3.8.2]. The positive elements in the JB-algebra $Symm(A)$ [**8**; 3.3.3] are called positive elements of A. The next lemma is the key tool in the proof of the desired LePage-type theorem for non-commutative JB^*-algebras.

LEMMA 3.2.- *Let A be a non-commutative JB^*-algebra, M a closed ideal of A, and λ be in $\mathbb{C}$. Then, for x, y in A, we have*

$$\begin{aligned} \| \lambda xy + (1-\lambda)yx + M \| \; &\leq \; \inf\{\| \lambda(x+m)y + (1-\lambda)y(x+m) \| \; : \; m \in M\} \\ &\leq \; (| \lambda | + | 1-\lambda |) \| \lambda xy + (1-\lambda)yx + M \| \; . \end{aligned}$$

Proof.- Let x, y be in A. Then, clearly, the first inequality in the statement holds. To prove the second inequality, let us fix $\epsilon > 0$. For elements u, v in any complex algebra containing A as a subalgebra, we put

$$u \square v := \lambda uv + (1-\lambda)vu \ .$$

Then we can choose q in M with

$$\| x\square y + q \| \quad \leq \quad \| x\square y + M \| + \epsilon \ .$$

We claim that there exists a positive element p in M satisfying $\| p \| \leq 1$ and

$$\| q - p\square q \| + \| p\square(x\square y) - (p\square x)\square y \| \quad < \ \epsilon \ .$$

Since M is $*$-invariant [**15**; Corollary 1.11] (hence a non-commutative JB^*-algebra), we can apply [**8**; Proposition 3.5.4] to the JB-algebra $Symm(M)$ to get a net $\{e_\lambda\}_{\lambda\in\Lambda}$ of positive elements of M such that $\| e_\lambda \| \leq 1$ for all λ in Λ, and $\lim\{e_\lambda.m\} = m$ for every m in M (where, as usual, the symbol "." stands for the product of A^+). On the other hand, the bidual A^{**} of A can be regarded as a non-commutative JB^*-algebra which enlarges A [**15**; Theorem 1.7] and whose product becomes separately w^*-continuous [**15**; Theorem 3.5]. Then the bipolar $M^{\circ\circ}$ of M in A^{**} is a w^*-closed ideal of A^{**}, and hence we have $M^{\circ\circ} = A^{**}e$ for some central $*$-invariant idempotent e in A^{**}[**15**; Theorem 3.9]. Since e is a unit for $M^{\circ\circ}$, it follows from the separate w^*-continuity of the product of A^{**} and the w^*-density of M in $M^{\circ\circ}$ that e is the unique possible w^*-cluster point of the net $\{e_\lambda\}$ in A^{**}. Since the closed unit ball of A^{**} is w^*-compact, we actually have that $w^* - \lim\{e_\lambda\} = e$. Now, note that the product $\square$ on A^{**} is separately w^*-continuous, and regard the space $A^{**} \times A^{**}$ as the bidual of the Banach space $A \times A$ with the sum norm. Then in $A^{**} \times A^{**}$ we have

$$\begin{aligned} &w^* - \lim\{(q - e_\lambda\square q, e_\lambda\square(x\square y) - (e_\lambda\square x)\square y)\} \\ &= (q - e\square q, e\square(x\square y) - (e\square x)\square y) = (0,0), \end{aligned}$$

where the last equality holds because e is a unit for $(M^{\circ\circ}, \square)$ and a central element of $(A^{**}, \square)$. Since the net $\{(q - e_\lambda\square q, e_\lambda\square(x\square y) - (e_\lambda\square x)\square y)\}$ lies in $A \times A$, it follows that $\{(q - e_\lambda\square q, e_\lambda\square(x\square y) - (e_\lambda\square x)\square y)\}$ converges to $(0,0)$ in the weak topology of $A \times A$, and therefore, for a suitable element p in the convex hull of the set $\{e_\lambda \ : \ \lambda \in \Lambda\}$, we have

$$\| q - p\square q \| + \| p\square(x\square y) - (p\square x)\square y \| \quad < \ \epsilon \ .$$

Clearly, such a p lies in M, is positive, and satisfies $\| p \| \leq 1$. Now that the claim is proved, recall that A^{**} has a unit $\mathbf{1}$ [**15**; Corollary 3.3] which is also a unit for $(A^{**}, \square)$, so that we can write

$$
\begin{aligned}
\inf\{\| (x+m)\Box y \| \ : \ m \in M\} &\leq \| (x - p\Box x)\Box y \| \\
&\leq \| x\Box y - p\Box(x\Box y) \| + \| p\Box(x\Box y) - (p\Box x)\Box y \| \\
&= \| (\mathbf{1}-p)\Box(x\Box y + q) - (\mathbf{1}-p)\Box q \| + \| p\Box(x\Box y) - (p\Box x)\Box y \| \\
&\leq (| \lambda | + | 1-\lambda |) \| x\Box y + q \| + \| q - p\Box q \| + \| p\Box(x\Box y) - (p\Box x)\Box y \| \\
&\leq (| \lambda | + | 1-\lambda |)(\| x\Box y + M \| + \epsilon) + \epsilon \ .
\end{aligned}
$$

By letting $\epsilon \to 0$, we obtain

$$\inf\{\| (x+m)\Box y \| \ : \ m \in M\} \leq (| \lambda | + | 1-\lambda |) \| x\Box y + M \| \ . \blacksquare$$

Taking $\lambda = 1$ in the above lemma, it follows that, if A is a non-commutative JB^*-algebra, if M is a closed ideal of A, and if x, y are in A, then the equality $\| xy + M \| = \inf\{\| (x+m)y \| \ : \ m \in M\}$ holds. In the proof of the following lemma, $M_2(\mathbb{C})$ will denote the algebra of all 2×2 complex matrices endowed with its natural structure of C^*-algebra (when it is identified with the algebra of all bounded linear operators on the two-dimensional complex Hilbert space).

LEMMA 3.3.- *Let A be a C^*-algebra such that there exist λ in $\mathbb{C}\backslash\{\frac{1}{2}\}$ and $k > 0$ satisfying $\| \lambda yx + (1-\lambda)xy \| \leq k \| \lambda xy + (1-\lambda)yx \|$ for all x, y in A. Then A is commutative.*

Proof.- Assume for the moment that A has a unit. If B denotes the complex algebra consisting of the vector space of A and the product $\Box$ given by $x\Box y := \lambda xy + (1-\lambda)yx$, then the unit of A is a unit for B, and, up to the multiplication of the norm of A by a suitable positive number, B becomes a complete normed algebra. Moreover, since $B^+ = A^+$, B is Jordan admissible and, for x in A, we have $Sp(A, x) = Sp(B, x)$. Now take z in A such that $z^2 = 0$. Since $Sp(B, z) = \{0\}$ and $\| y\Box x \| \leq k \| x\Box y \|$ for all x, y in B, it follows from Theorem 2.2 that z $\Box$-commutes with every element of B. Applying that $\lambda \neq \frac{1}{2}$, we find that z commutes (in the usual sense) with every element of A, in particular with z^*. Therefore we have $\| z \|^4 = \| z^*z \|^2 = \| (z^*z)^2 \| = \| z^2(z^*)^2 \| = 0$, hence $z = 0$. Keeping in mind Kaplansky's theorem, the proof would be concluded in the unital case. However, since the consideration of the non unital situation will need a refined version of Kaplansky's theorem proved in [**9**], we remove the incidental assumption that A has a unit, and limit ourselves to codify a straightforward consequence of the above argument. Indeed, since the condition

$$\| \lambda yx + (1-\lambda)xy \| \leq k \| \lambda xy + (1-\lambda)yx \| \ \text{ for all } x, y \text{ in } A$$

is inherited by any subalgebra of A, and the C^*-algebra $M_2(\mathbb{C})$ has a unit as well as non zero elements z with $z^2 = 0$, A cannot contain $M_2(\mathbb{C})$ as a C^*-subalgebra. According to [**9**; Corollary 9], to conclude the proof it is enough to show that

A also cannot contain as a C^*-subalgebra the C^*-algebra $\mathcal{C}_0([0,1], M_2(\mathbb{C}))$ of all continuous mappings from $[0,1]$ to $M_2(\mathbb{C})$ vanishing at zero. Assume by the contrary that $\mathcal{C}_0([0,1], M_2(\mathbb{C}))$ is a C^*-subalgebra of A, so that the inequality

$$\| \lambda yx + (1-\lambda)xy \| \;\; \leq \;\; k \| \lambda xy + (1-\lambda)yx \|$$

is true for all x, y in $\mathcal{C}_0([0,1], M_2(\mathbb{C}))$. Put

$$M := \{x \in \mathcal{C}_0([0,1], M_2(\mathbb{C})) \;:\; x(1) = 0\}.$$

Since M is a closed ideal of $\mathcal{C}_0([0,1], M_2(\mathbb{C}))$, we can apply Lemma 3.2 to obtain

$$\begin{aligned}\| \lambda yx + (1-\lambda)xy + M \| \;\; &\leq \;\; \inf\{\| \lambda y(x+m) + (1-\lambda)(x+m)y \| \;\; : \; m \in M\} \\ &\leq \; k \inf\{\| \lambda(x+m)y + (1-\lambda)y(x+m) \| \;\; : \; m \in M\} \\ &\leq \; k(| \lambda | + | 1-\lambda |) \| \lambda xy + (1-\lambda)yx + M \|\end{aligned}$$

for all x, y in $\mathcal{C}_0([0,1], M_2(\mathbb{C}))$. Since $\mathcal{C}_0([0,1], M_2(\mathbb{C}))/M$ is isometrically isomorphic to $M_2(\mathbb{C})$, we deduce that there exists $k' := k(| \lambda | + | 1-\lambda |) > 0$ satisfying $\| \lambda yx + (1-\lambda)xy \| \leq k' \| \lambda xy + (1-\lambda)yx \|$ for all x, y in $M_2(\mathbb{C})$. But we have seen in the first part of the proof that such a situation cannot happen. ■

THEOREM 3.4.- *Let A be a non-commutative JB^*-algebra such that there exist λ in $\mathbb{C}\backslash\{\frac{1}{2}\}$ and $k > 0$ satisfying $\| \lambda yx + (1-\lambda)xy \| \leq k \| \lambda xy + (1-\lambda)yx \|$ for all x, y A. Then A is commutative.*

Proof.- As we have seen in the proof of Lemma 3.3, Lemma 3.2 implies that, if M is a closed ideal of A, and if α, β are in A/M, then we have

$$\| \lambda\beta\alpha + (1-\lambda)\alpha\beta \| \;\; \leq \;\; k' \| \lambda\alpha\beta + (1-\lambda)\beta\alpha \| \;\; ,$$

where $k' := k(| \lambda | + | 1-\lambda |)$. Now, the structure theory for non-commutative JB^*-algebras (see [**15**; Lemma 5.3 and Theorem 5.4] and [**16**; Corollary 1.13 and Theorems 2.7 and 3.2]), together with the facts that quotients of non-commutative JB^*-algebras are non-commutative JB^*-algebras [**15**; Corollary 1.11] and that injective *-homomorphisms between non-commutative JB^*-algebras are isometries [**23**], provides us with a family $\{M_i\}_{i\in I}$ of closed ideals of A (namely, the kernels of the so-called type I factor representations of A) satisfying the following two properties:

1. $\cap_{i\in I} M_i = 0$.
2. If, for i in I, the algebra A/M_i is neither commutative nor quadratic, then the next situation occurs:

$(\mathcal{S}_i)$ There exists a C^*-algebra A_i such that $A/M_i = A_i$ as involutive Banach spaces, and the product of A/M_i is related to that of A_i (denoted by $\circ$, say) by means of the equality $\alpha\beta = \mu_i \alpha \circ \beta + (1-\mu_i)\beta \circ \alpha$, where μ_i is a fixed real number with $\frac{1}{2} < \mu_i \leq 1$.

Let i be in I enjoying the situation $(\mathcal{S}_i)$. Putting

$$\lambda_i := \frac{1}{2} + 2(\lambda - \frac{1}{2})(\mu_i - \frac{1}{2}),$$

for α, β in A/M_i we have

$$\lambda\alpha\beta + (1-\lambda)\beta\alpha = \lambda_i \alpha \circ \beta + (1-\lambda_i)\beta \circ \alpha ,$$

and hence $\| \lambda_i \beta \circ \alpha + (1-\lambda_i)\alpha \circ \beta \| \leq k' \| \lambda_i \alpha \circ \beta + (1-\lambda_i)\beta \circ \alpha \|$. Since $(A_i, \circ)$ is a C^*-algebra and $\lambda_i \neq \frac{1}{2}$, it follows from Lemma 3.3 that A_i is commutative. Then the relation between the products of A/M_i and A_i shows that both products coincide, and hence A/M_i is commutative too.

Now, let i be in I such that A/M_i is a quadratic algebra. If B_i denotes the complex algebra consisting of the vector space of A/M_i and the product $\Box$ given by $\alpha \Box \beta := \lambda\alpha\beta + (1-\lambda)\beta\alpha$, then, up to the multiplication of the norm of A/M_i by a suitable positive number, B_i becomes a complete normed quadratic algebra. Since the inequality $\| \beta \Box \alpha \| \leq k' \| \alpha \Box \beta \|$ holds for all α, β in B_i, it follows from Corollary 2.3 that B_i is commutative. Since $\lambda \neq \frac{1}{2}$, A/M_i is commutative too.

From the last two paragraphs and Property 2 of the family $\{M_i\}_{i \in I}$ it follows that A/M_i is commutative for every i in I. Finally, from Property 1 of that family we deduce that A is commutative. ■

COROLLARY 3.5.- *Let A be a non-commutative JB^*-algebra such that there exists $k > 0$ satisfying $\| yx \| \leq k \| xy \|$ for all x, y in A. Then A is commutative.*

REMARK 3.6.- Non-commutative JBW^*-algebras are defined as those non-commutative JB^*-algebras which are dual Banach spaces. If A is a non-commutative JBW^*-algebra, then A has a unit [**15**; Corollary 3.3] and is the closed linear hull of its *-invariant idempotents (indeed, $Symm(A)$ is a JBW-algebra [**4**], and [**8**; Proposition 4.2.3] applies). Since idempotents have a finite spectrum, when A is actually a non-commutative JBW^*-algebra Corollary 3.5 follows directly from Theorem 2.2. The same comment applies to Theorem 3.4 (indeed, apply Theorem 2.2 to the algebra obtained by replacing the product of A by the one $(x, y) \to \lambda xy + (1-\lambda)yx$). It is also worth mentioning that, if A is a (commutative) JBW^*-algebra, and if e denotes the unit of A, then, by Corollary 2.1, every left e-admissible product $\odot$ on A satisfying $\| x \odot y \| \leq k \| xy \|$ for some positive constant k and all x, y in A must coincide with the canonical product of A.

4.- Discussion of results and methods

Let us recall the main question we have dealt with.

PROBLEM 4.1.- *Let A be a complete normed complex algebra with a unit, and assume that there exists a positive constant k satisfying $\| yx \| \leq k \| xy \|$ for all x, y in A. Must A be commutative?*

More ambitious questions are the following.

PROBLEM 4.2.- *Let A be a complete normed complex algebra with a right unit e, and $\odot$ be a right e-admissible product on A satisfying*

$$\| x \odot y \| \leq k \| xy \|$$

for some positive constant k and all x, y in A. Does $\odot$ coincide with the canonical product of A?

PROBLEM 4.3.- *Let A be a complete normed complex algebra with a right unit e, E a complex Banach space, and $h : A \times A \to E$ a bilinear mapping satisfying $\| h(x, y) \| \leq k \| xy \|$ for some positive constant k and all x, y in A. Does the equality $h(x, y) = h(xy, e)$ holds for every x, y in A?*

According to LePage's argument and the results in this paper, the answer to Problem 4.3 (and hence also to Problems 4.2 and 4.1) is affirmative if A is either associative or finite-dimensional. Therefore, most probably, the answer must remain affirmative without any additional requirement. Actually, LePage's argument and our techniques share a common idea, which we explain in what follows.

Let A be a complete normed complex algebra A with a right unit e. For z in A, consider the following property

($\mathcal{P}_z$) *There exists a couple (Ω, φ), where Ω is the complement in $\mathbb{C}$ of a countable compact set such that $0 \in \Omega$, and $\varphi : \Omega \to A$ is an analytic mapping satisfying:*

1. *$\varphi(0) = e$.*
2. *$\varphi'(0) = z$.*
3. *The operator $R_{\varphi(\lambda)}$ is bijective for every λ in Ω.*

Then, looking at the proof of Theorem 1.1, we realize that $(\mathcal{P}_z)$ holds whenever $Sp(R_z)$ is countable and $0 \notin Sp(R_z)$, and that Problem 4.3 has an affirmative answer whenever A is the closed linear hull of the set

$$\{z \in A : z \text{ satisfies } (\mathcal{P}_z)\}.$$

Thus, Problem 4.3 answers affirmatively in the finite-dimensional case. If A is associative, then Problem 4.3 answers affirmatively because every element z in A satisfies the improved version of $(\mathcal{P}_z)$ given by

$(\mathcal{P}_z^*)$ *There exists an analytic function $\varphi : \mathbb{C} \to A$ such that $\varphi(0) = e$, $\varphi'(0) = z$, and the operator $R_{\varphi(\lambda)}$ is bijective for every λ in $\mathbb{C}$.*

Indeed, when A is associative and z is in A, the analytic mapping $\varphi : \mathbb{C} \to A$ defined by

$$\varphi(\lambda) := e + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} z^n$$

satisfies all the requirements in $(\mathcal{P}_z^*)$ (note that, thanks to the equality $R_{\varphi(\lambda)} = \exp(\lambda R_z)$, $R_{\varphi(\lambda)}$ is certainly a bijective operator for every λ in $\mathbb{C}$). A similar privilege situation happens in the more general case that A is right alternative (i.e., the equality $yx^2 = (yx)x$ holds for all x, y in A), as we see in the sequel. For x in such an algebra A, the right alternative identity reads as $R_{x^2} = (R_x)^2$, so that, after linearization, we obtain $R_{xy+yx} = R_x R_y + R_y R_x$ for all x, y in A. Now take z in A, and define a sequence $\{z_n\}$ in A by $z_1 = z$ and $z_{n+1} = \frac{1}{2}(zz_n + z_n z)$. It follows from an elementary induction that $R_{z_n} = (R_z)^n$ for every n in $\mathbb{N}$, and hence the analytic mapping $\varphi : \mathbb{C} \to A$ defined by

$$\varphi(\lambda) := e + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} z_n$$

satisfies $\varphi(0) = e$, $\varphi'(0) = z$, and $R_{\varphi(\lambda)} = \exp(\lambda R_z)$. Therefore we can formulate the result which follows.

PROPOSITION 4.4.- *Problem 4.3 has an affirmative answer whenever A is right alternative.*

The following example shows that the privilege situation for the property $(\mathcal{P}_z)$ occurring in the right alternative setting cannot be expected in general.

EXAMPLE 4.5.- Let A be the unital Jordan complex algebra whose vector space is $\mathbb{C}^3$ and whose product is defined by

$$(x_1, x_2, x_3)(y_1, y_2, y_3) := (\ x_1y_1 + x_2y_2\ ,\ x_1y_2 + x_2y_1,\ x_1y_3 + x_3y_1\ )\ .$$

A straightforward calculation shows that, for $x = (x_1, x_2, x_3)$ in A, the equality $\det(R_x) = x_1(x_1^2 - x_2^2)$ holds. Let us fix $z = (z_1, z_2, z_3)$ in A satisfying $(\mathcal{P}_x^*)$, so that there are complex valued entire functions $\varphi_1, \varphi_2, \varphi_3$ satisfying $\varphi_1(0) = 1$, $\varphi_2(0) = \varphi_3(0) = 0$, $\varphi_i'(0) = z_i$ for $i = 1, 2, 3$, and $\varphi_1(\lambda)(\varphi_1(\lambda)^2 - \varphi_2(\lambda)^2) \neq 0$ for all λ in $\mathbb{C}$. Since $\varphi_1(\lambda) \neq 0$ for every λ in $\mathbb{C}$, and the mapping $\lambda \to \frac{\varphi_2(\lambda)}{\varphi_1(\lambda)}$

is an entire function which does not take the values 1 and -1, it follows from Picard's theorem [**20**; Theorem 16.22] that there exists a constant c such that $\varphi_2(\lambda) = c\varphi_1(\lambda)$ for every λ in $\mathbb{C}$. Now, we have $c = c\varphi_1(0) = \varphi_2(0) = 0$, so $\varphi_2 = 0$, and so $z_2 = \varphi_2'(0) = 0$. In this way, the (closed) linear hull of the set

$$\{z \in A \ : \ z \text{ satisfies } (\mathcal{P}_z^*)\}$$

is not the whole algebra A.

According to the above example, the refinement of LePage's argument made in this paper (by considering Property $(\mathcal{P}_z)$ instead of $(\mathcal{P}_z^*)$ becomes crucial when we want to remove associativeness in the classical results.

Now that we have discussed about the proof of Theorem 1.1, let us do the same in relation to Theorem 2.2. We begin by noting that the key idea in the proof of the last quoted theorem is nothing but a simplified version of the following claim.

CLAIM 4.6.- *Let A be a complete normed complex algebra with a right unit e, let $\odot$ be a right e-admissible product on A such that the inequality*

$$\| x \odot y \| \leq k \| xy \|$$

holds for some positive constant k and all x, y in A, and let z be in A satisfying $(\mathcal{P}_z)$ (respectively $(\mathcal{P}_z^*)$ relative to the product $\odot$. Then z satisfies $(\mathcal{P}_z)$ (respectively $(\mathcal{P}_z^*)$ relative to the canonical product of A.

Proof.- Chose a couple (Ω, φ), where Ω is the complement in $\mathbb{C}$ of a countable compact set satisfying $0 \in \Omega$, and $\varphi : \Omega \to A$ is an analytic mapping such that $\varphi(0) = e$, $\varphi'(0) = z$, and the operator $R^{\odot}_{\varphi(\lambda)}$ is bijective for every λ in Ω. Then, for every x in A and every λ in Ω, we have

$$\| x \| \ \leq \ \| (R^{\odot}_{\varphi(\lambda)})^{-1} \| \| R^{\odot}_{\varphi(\lambda)}(x) \| \ \leq \ k \| (R^{\odot}_{\varphi(\lambda)})^{-1} \| \| R_{\varphi(\lambda)}(x) \| \ .$$

Therefore, for every λ in Ω, the operator $R_{\varphi(\lambda)}$ is bounded below. Now consider the set

$$\Omega' := \{\lambda \in \Omega \ : \ R_{\varphi(\lambda)} \text{ is bijective}\} \ ,$$

and assume that $\Omega' \neq \Omega$. Then, since Ω' is non empty (indeed, $R_{\varphi(0)}$ is the identity mapping on A) and Ω is connected, there must exist some λ_0 in the boundary of Ω' relative to Ω. For such a λ_0, $R_{\varphi(\lambda_0)}$ lies in the boundary of the set of all invertible elements of $BL(A)$, and hence, by [**1**; Lemma 56.3 and Theorem 57.4], it is not bounded below. This is a contradiction. ■

The claim just proved, together with the previous discussion about the proof of Theorem 1.1, leads to the next result.

PROPOSITION 4.7.- *Problem 4.2 has an affirmative answer whenever the product $\odot$ is right alternative.*

Given a complex algebra A and a complex number λ, the λ-mutation of A is defined as the algebra consisting of the vector space of A and the product

$$(x, y) \ \rightarrow \ \lambda xy + (1 - \lambda)yx \ .$$

Note that, if the algebra A has a unit e, then e remains a unit in any mutation of A.

COROLLARY 4.8.- *Problem 4.1 has an affirmative answer if A has a right alternative mutation (for instance, if A^+ is associative).*

Proof.- By assumption, there exists λ in $\mathbb{C}$ such that the product $\odot$ on A defined by $x \odot y := \lambda xy + (1-\lambda)yx$ is right alternative. If $\lambda = 1$, then the result follows from Proposition 4.4. Otherwise, since for x, y in A we have

$$\| x \odot y \| \ \leq \ (\mid \lambda \mid + k \mid 1 - \lambda \mid) \| xy \| \ ,$$

the result follows from Proposition 4.7. ■

A complex algebra is said to be split quasiassociative if it is a mutation of a complex associative algebra.

COROLLARY 4.9.- *Problem 4.1 has an affirmative answer whenever A is split quasiassociative.*

Proof.- Choose an associative product $\odot$ on A and λ in $\mathbb{C}$ satisfying

$$xy \ = \ \lambda x \odot y + (1 - \lambda) y \odot x$$

for all x, y in A. If $\lambda = \frac{1}{2}$, then A is obviously commutative. Otherwise, putting $\mu := \lambda(2\lambda - 1)^{-1}$, we have

$$x \odot y \ = \ \mu xy + (1 - \mu) yx$$

for all x, y in A. Therefore A has an associative mutation, and Corollary 4.8 applies. ■

REMARK 4.1O.- A prime complex algebra A is said to be centrally closed if, for every non zero ideal M of A and for every linear mapping $f : M \rightarrow A$ satisfying $f(ax) = af(x)$ and $f(xa) = f(x)a$ for all x in M and a in A, there exists λ in $\mathbb{C}$ such that $f(x) = \lambda x$ for all x in M. According to the main result in [**22**], if A is a centrally closed prime nondegenerate non-commutative Jordan complex algebra, then at least one of the following assertions hold:

1. A is commutative.
2. A is quadratic.
3. A^+ is associative.
4. A is split quasiassociative.

Now, it follows from Corollaries 2.3, 4.8, and 4.9 that Problem 4.1 has an affirmative answer if A is a centrally closed prime non degenerate non-commutative Jordan algebra. We note that complete normed primitive non-commutative Jordan complex algebras are prime, nondegenerate, and centrally closed [**19**].

With the above remark, the discussion of the proof of Theorem 2.2 is concluded.

Now, let us point out that the original LePage's technique can be easily adapted to provide further interesting developments in the non-associative setting. A first sample of this procedure is shown in the following proposition. Recall that an algebra A is called power-associative if every one-generated subalgebra of A is associative. Non-commutative Jordan algebras are examples of power-associative algebras [**21**; p. 141].

PROPOSITION 4.11.- *Let A be a complete normed power-associative complex algebra with a unit. Then A is associative if (and only if) there exists $k > 0$ satisfying $\| x(yz) \| \leq k \| (xy)z \|$ for all x, y, z in A.*

Proof.- For λ in $\mathbb{C}$ and x, y, z in A, we have

$$\| \exp(\lambda x)[(\exp(-\lambda x)y)z] \| \;\leq\; k \| [\exp(\lambda x)(\exp(-\lambda x)y)]z \|$$

$$\leq\; k \| \exp(\lambda x)(\exp(-\lambda x)y) \| \| z \| \;\leq\; k^2 \| y \| \| z \| \;.$$

Therefore the analytic mapping

$$\lambda \;\rightarrow\; \exp(\lambda x)[(\exp(-\lambda x)y)z] \;=\; yz \;+\lambda[x(yz) - (xy)z] \;+\; ...$$

from $\mathbb{C}$ to A is bounded, and hence constant. It follows $x(yz) - (xy)z = 0$. ■

Other non-associative applications of LePage's technique follow from the next general result.

PROPOSITION 4.12.- *Let A be a complete normed non-commutative Jordan complex algebra with a unit e, and $P : A \rightarrow BL(A)$ a quadratic mapping such that $P_e = \mathbf{1}$ (the identity mapping on A) and $\| P_x(y) \| \leq k \| U_x(y) \|$ for some $k > 0$ and all x, y in A. Then $P = U$.*

Proof.- For λ in $\mathbb{C}$ and x in A, $\exp(\lambda x)$ is an invertible element of A, and hence $U_{\exp(\lambda x)}$ is a bijective operator [**10**; Theorem 13, p. 52], so that, by the assumed inequality, we have $\| P_{\exp(\lambda x)} \circ U^{-1}_{\exp(\lambda x)} \| \leq k$. Let x be in A. Then the mapping $\lambda \to P_{\exp(\lambda x)} \circ U^{-1}_{\exp(\lambda x)}$ from $\mathbb{C}$ to $BL(A)$ is analytic and bounded, and hence the equality

$$P_{\exp(\lambda x)} = U_{\exp(\lambda x)}$$

holds for every λ in $\mathbb{C}$ (since $P_e = \mathbf{1}$). Now, computing first and second derivatives at $\lambda = 0$, and combining the two resulting equalities, we find $P_x = U_x$. ■

Recall that an algebra A is called alternative if the identities $x^2 y = x(xy)$ and $yx^2 = (yx)x$ hold for all x, y in A. Actually, in an alternative algebra, all two-generated subalgebras are associative [**21**; p. 29]. By taking in Proposition 4.12 $P_x(y) := xyx$, $P_x(y) := x^2 y$, and $P_x(y) := 2x(xy) - x^2 y$, and applying well-known identities in non-commutative Jordan and alternative algebras, we find Assertions 1, 2, and 3, respectively, in the corollary which follows.

COROLLARY 4.13.- *Let A be a complete normed non-commutative Jordan complex algebra with a unit. Then we have:*

1. *A is alternative if (and only if) there exists $k > 0$ satisfying*

$$\| xyx \| \leq k \| U_x(y) \|$$

   for all x, y in A.

2. *A is associative and commutative if (and only if) there exists $k > 0$ satisfying $\| x^2 y \| \leq k \| U_x(y) \|$ for all x, y in A.*

3. *A is commutative if (and only if) there exists $k > 0$ satisfying*

$$\| 2x(yx) - x^2 y \| \leq k \| U_x(y) \|$$

   for all x, y in A.

We conclude this section with an easy observation providing a non-unital LePage's type result for some normed alternative algebras. Let A be a real or complex alternative algebra. For x in A, the operator U_x has a very simple form, namely we have $U_x(y) = xyx$ for every y in A. Moreover A is non-degenerate (i.e., $U_x = 0$ implies $x = 0$) if (and only if) it is semiprime (i.e., if M is an ideal of A, and if $M^2 = 0$, then $M = 0$) [**24**; Theorem 9.2.5]. Now, assume that the alternative algebra A is normed. Then the condition

$$m \| x \|^2 \leq \| U_x \| \text{ for some positive constant } m \text{ and every } x \text{ in } A \qquad (\#)$$

becomes a natural analytic strengthening of semiprimeness. It is easily shown that the above condition is equivalent to the fact that every normed ultrapower of A is semiprime. Therefore we say that the normed alternative algebra A is ultra-semiprime whenever A satisfies (#).

OBSERVATION 4.14.- *Let A be an ultra-semiprime normed alternative complex algebra such that there exists $k > 0$ satisfying $\| yx \| \leq k \| xy \|$ for all x, y in A. Then A is commutative.*

Proof.- For x, y in A we have

$$\| U_x(y) \| \;=\; \| xyx \| \;\leq\; k \| x^2 y \| \;\leq\; k \| x^2 \| \| y \| \;,$$

and hence $\| U_x \| \;\leq\; k \| x^2 \|$. Now let $m > 0$ be such that $m \| x \|^2 \;\leq\; \| U_x \|$ for every x in A. Then, again for every x in A, the inequality $m \| x \|^2 \;\leq\; k \| x^2 \|$ holds. By [**17**; Proposition 31], A is commutative (note that the assumption in [**17**] that A has a unit is unnecessary). ■

Looking at the proof of Observation 3.1 we see that, if A is a C^*-algebra, then for every x in A we have $\| U_x \| = \| x \|^2$, and therefore C^*-algebras are ultra-semiprime. The same is true in the more general case of the so-called alternative C^*-algebras, for which the reader is referred to [**3**] and [**15**]. Since alternative C^*-algebras are non-commutative JB^*-algebras [**15**; Proposition 1.3], Observation 4.14 provides us with a very easy proof of Corollary 3.5 in the particular case that A is an alternative C^*-algebra.

5.- A refinement of LePage's associative theorem

This concluding section is devoted to Banach algebras (i.e., complete normed associative algebras). Our approach begins with the next proposition.

PROPOSITION 5.1.- *Let A be a complex Banach algebra, E a complex Banach space, and $h : A \times A \to E$ a bilinear mapping satisfying*

$$\| h(x, y) \| \leq k \| xy \|$$

for some positive constant k and all x, y in A. Then, for all x, y, z in A, we have $h(xy, z) \;=\; h(x, yz)$.

Proof.- Let x, y, z be in A. For λ in $\mathbb{C}$, we can consider $\exp(\lambda y)$ and $\exp(-\lambda y)$ as elements of the unital hull of A, so that $x \exp(\lambda y)$ and $\exp(-\lambda y) z$ lie in A. Therefore we have $\| h(x \exp(\lambda y), \exp(-\lambda y) z) \| \leq k \| xz \|$. The Liouville theorem leads to $h(x \exp(\lambda y), \exp(-\lambda y) z) = h(x, z)$, and hence $h(xy, z) - h(x, yz) = 0$ (by computing derivatives at $\lambda = 0$). ■

Taking in Proposition 5.1 $E = A$ and $h(x, y) = yx$, we obtain:

COROLLARY 5.2.- *Let A be a complex Banach algebra satisfying*

$$\| yx \| \leq k \| xy \|$$

for some positive constant k and all x, y in A. Then A^2 is contained in the centre $C(A)$ of A.

Let A be as in the above corollary. It follows that, if A^2 is dense in A, then A is commutative. Another not so clear consequence is provided by the next corollary. The annihilator $Ann(A)$ of an algebra A is defined as the set of those elements x in A satisfying $xA = Ax = 0$.

COROLLARY 5.3.- *Let A be a complex Banach algebra satisfying*

$$\| yx \| \leq k \| xy \|$$

for some positive constant k and all x, y in A. If $Ann(A) = 0$, then A is commutative.

Proof.- By Corollary 5.2, it is enough to show that the conditions $A^2 \subseteq C(A)$ and $Ann(A) = 0$ imply that A is commutative. But, applying the first condition, for x, y, z, t in A we have

$$0 = [xyz, t] = xy[z, t] + [xy, t]z = xy[z, t]$$

(here $[.,.]$ stands for the commutator on A). Therefore

$$[A, A]A^2 = A[A, A]A = A^2[A, A] = 0 \ ,$$

and hence $[A, A]A$ and $A[A, A]$ are contained in $Ann(A)$. Now, applying twice the second condition, it follows $[A, A] = 0$. ■

After Corollary 5.3, we realize that Observation 4.14 is not an interesting fact when applies in particular to associative algebras. However, the general assertion made in that observation for alternative algebras is independent of the above corollary. This is so because, as the next example shows, Corollary 5.3 does not remain true if the associativeness of A is relaxed to the alternativeness.

EXAMPLE 5.4.- Let A be the complex algebra whose vector space is $\mathbb{C}^7$ and whose product is defined by

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)(y_1, y_2, y_3, y_4, y_5, y_6, y_7) :=$$

$$(0, 0, 0, x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_2y_3 - x_3y_2, x_1y_6 - x_6y_1 + x_5y_2 - x_2y_5 + x_3y_4 - x_4y_3).$$

Then A is alternative and the equality $Ann(A) = 0$ holds. Moreover, for every algebra norm $\| \cdot \|$ on A and all x, y in A we have $\| xy \| = \| yx \|$ (since A is anticommutative).

The algebra in the above example also shows that in Corollary 5.2 the associativeness of A cannot be relaxed to the alternativeness. Indeed, in the alternative anticommutative algebra A of the example we have

$$(0,0,1,0,0,0,0)((1,0,0,0,0,0,0)(0,1,0,0,0,0,0)) = (0,0,0,0,0,0,1) \ ,$$

and therefore elements of A^2 need not commute with all elements of A. This remark is far from being anecdotist because every counterexample to the alternative generalisation of Corollary 5.3 must be also a counterexample to the alternative generalisation of Corollary 5.2. This follows from the fact (not too easy to show) that, if A is an alternative algebra over a field of characteristic different from 2 and 3, if $Ann(A) = 0$, and if every element in A^2 commutes with all elements of A, then A is commutative.

Let $\mathcal{L}$ denote the class of complex Banach algebras A satisfying

$$\| yx \| \leq k \| xy \|$$

for some positive constant k and all x, y in A. If a complex Banach algebra A is the direct sum of a closed commutative ideal and a closed anticommutative ideal, then certainly A is a member of $\mathcal{L}$. However, as the following example shows, there are members of $\mathcal{L}$ of a more complicated nature.

EXAMPLE 5.5.- Consider the associative complex algebra A whose vector space is $\mathbb{C}^5$ and whose product is defined by

$$(x_1, x_2, x_3, x_4, x_5)(y_1, y_2, y_3, y_4, y_5) := (x_2 y_3 - y_2 x_3, 0, 0, 0, x_2 y_4 + y_2 x_4) \ .$$

Then A (endowed with any algebra norm) is a member of $\mathcal{L}$. Moreover, it is easily seen that A cannot be expressed as a direct sum of a commutative ideal and an anticommutative ideal.

The next example shows that the necessary condition $A^2 \subseteq C(A)$ for A to be a member of $\mathcal{L}$, provided by Proposition 5.1, is far from being sufficient.

EXAMPLE 5.6.- Given λ in $\mathbb{C}$, consider the associative complex algebra A whose vector space is $\mathbb{C}^3$ and whose product is defined by

$$(x_1, x_2, x_3)(y_1, y_2, y_3) := (0, 0, x_1 y_2 + \lambda x_2 y_1) \ .$$

Since $A^3 = 0$, certainly the inclusion $A^2 \subseteq C(A)$ holds. Assume that for some algebra norm $\| \ . \ \|$ on A there is $k > 0$ satisfying $\| yx \| \leq k \| xy \|$ whenever x, y are in A. Then, for all complex numbers x_1, x_2, y_1, y_2 we have

$$| \ y_1 x_2 + \lambda y_2 x_1 \ | \leq k \ | \ x_1 y_2 + \lambda x_2 y_1 \ | \ \ .$$

Taking $x_1 = x_2 = y_1 = 1$ and $y_2 = -\lambda$, we obtain $1 - \lambda^2 = 0$, and hence $\lambda = \mp 1$.

We conclude the paper by noting that some results in Section 4 can be reformulated in terms of Banach algebras. Indeed, Corollary 4.9 ensures that, if A is a complex Banach algebra with a unit, and if there exist $\lambda \in \mathbb{C}\backslash\{\frac{1}{2}\}$ and $k > 0$ satisfying $\| \lambda yx + (1-\lambda)xy \| \leq k \| \lambda xy + (1-\lambda)yx \|$ for all x, y in A, then A is commutative. Also, as a consequence of Corollary 4.8 we obtain that, if A is a commutative complex Banach algebra with a unit e, and if $\odot$ is a continuous anticommutative product on A satisfying $x \odot e = 0$ and $\| xy - x \odot y \| \leq k \| xy + x \odot y \|$ for some $k > 0$ and all x, y in A, then the product $\odot$ is identically zero.

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